Optimal pricing of successive generations of product advances

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We obtain the optimal pricing policy of a monopolist marketing successive generations of product advances. We characterize the impact of foresight regarding entry of the second generation product and the technological substitution and market expansion created by the introduction of the second generation product on the optimal pricing policy of the monopolist. We also obtain the optimal pricing policies of the producers in situations where the successive generations of product advances are marketed by independent producers. By contrasting these policies with those of the monopolist marketing successive generations of product advances, we provide insight on the differential impact of foresight, technological substitution and market expansion on the optimal pricing policies of the producers.

1. Introduction

The high-technology product market of today is characterized by waves of new product introductions and improvements. The memory chip market with introduction of the 64K, 256K and 1M VRAM chip in 1983, 1986 and 1988, respectively, the math processor market with new product introductions in 1985, 1987 and 1989, respectively, are but two examples of this phenomenon. The rapid pace of technological change and the overlapping product life cycles that characterize these markets generate idiosyncratic effects that substantially differentiate these markets from conventional markets. For example, the introduction of the next-generation product presents potential buyers of the earlier technology with an opportunity to switch to the newer technology (also referred to as the technological substitution or cannibalization effect). The newer technology, by virtue of its greater application potential, also expands the market potential by opening up new markets (also referred to as the market expansion effect). From the firm's perspective, knowledge regarding introduction of a new product in the near future (referred to as foresight) and the technological substitution and market expansion effects

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created by it can be expected to have a significant impact on the strategy pursued by the firm with regard to these products.

The extant literature in marketing by generally focusing on issues related to marketing of a single product has little to say on the implications of these effects for optimal policies to be followed by firms in these situations. Growing recognition of the importance of these markets (see Mahajan et al., 1990) has spurred researchers in recent years to explore some of these issues (see Mahajan and Muller, 1991; Bayus, 1992). We seek to complement the burgeoning literature in this area by analyzing the pricing policy implications of these idiosyncratic effects for firms marketing successive generations of product advances.

In particular, we are interested in answers to the following questions. What is the impact of technological substitution and market expansion on the optimal pricing policy of a firm producing these products? What is the effect of foresight on the pricing strategy adopted by the firm with regard to the older technology? How does the impact of these effects on the optimal pricing strategy vary depending on whether the same firm markets the successive generations of product advances or different firms market the successive generations?

The rest of the paper is organized as follows. We provide a brief review of the relevant literature in this area in Section 2. In Section 3 we present the model and obtain the optimal pricing policies for a monopolist marketing successive generations of product advances. We contrast these policies with those adopted by independent producers of the successive generations of product advances in Section 4. We conclude in Section 5 with a brief summary and discussion.

2. Background literature

There exists a considerable literature in marketing on optimal pricing policy. The interested reader is referred to Kalish (1988) for an exhaustive review. However, most of this research has focused on obtaining the optimal price policy for a firm producing a single product. Norton and Bass (1987) develop a diffusion model that successfully captures the technological substitution and market expansion effects that characterize markets with successive generations of product advances. However, as in the Bass (1969) diffusion model for a single product, their model ignores the impact of marketing mix variables. From a marketing policy perspective, the exclusion of marketing mix variables is of particular concern in this context since decisions on a marketing mix variable with respect to one product have a significant impact on all the other products in the product mix.

Researchers in recent years have started investigating problems of this nature. Mahajan and Muller (1991) analyze the pricing problem in the context of a market for primary and contingent products. They show that an integrated monopolist who produces both products will price both (the primary and contingent) products lower than independent firms that produce each of these product separately. Furthermore, they show that contrary to conventional wisdom, the firm makes more money on the primary product as opposed to the contingent product. Bayus (1992) in his paper analyzes the impact of replacement behavior of consumers of durables on the pricing strategy of a firm producing successive generations of consumer durables. He provides in his paper an explanation for the generally observed phenomenon of declining prices over time based on buying behavior of consumer for products with overlapping replacement cycles.
This paper explores the pricing policy implications for producers of successive generations of product advances. It complements the work of Bayus (1992) by allowing for market expansion effects created by introduction of the newer technology and by incorporating the impact of foresight on the pricing strategy of the firm. It complements the work of Mahajan and Muller (1991) by providing insight on the differential effect of foresight with regard to introduction of a newer technology on the pricing strategies of the integrated and independent producers.

3. The model

Consider a monopolist marketing successive generations of product advances to consumers who differ in their evaluation of the value of the products. The consumers combine to form an aggregate demand curve. The objective of the monopolist is to maximize profits from the product line over the planning horizon \([0, T]\). While the monopoly assumption may seem restrictive, it is not unfamiliar in the technology product market. Reasons including but not necessarily restricted to patent protection, proprietary technology, etc. are responsible for this market structure. Intel, which has enjoyed a virtual monopoly in the microprocessor market with its 4040, 8080, 8086 and more recently the 8088 line of products, is a good example.

For the sake of easier exposition, we will assume that the firm markets only two products over the planning horizon. We will assume without loss of generality that the first-generation product has been launched at the beginning of the planning horizon (i.e., at time \(t = 0\)). The point of departure of our research from the existing literature is our consideration of the firm's decision to introduce an advanced version of the product at time \(t = E\) (where \(E \in (0, T)\)). From time \(t = E\) onward, the firm sells both products. The entry time \(E\) is specified exogenously and is not a decision variable in the model. The sales scenario is as sketched in Fig. 1. Our decision to treat the entry time \(E\) as fixed and exogenous is based partly on the information obtained from conversations with practitioners. They indicate that managers in

![Conceptual model](image-url)
reality have limited flexibility with regard to the decision on the entry time of a new product. The limited flexibility is a consequence of the constraints placed on the manager by a variety of factors ranging from delivery schedule of the R&D department, testing requirements for the product, manufacturing schedules, lead time for marketing activities and the demands of channel members. The assumptions is also based partly on the fact that the problem we are interested in is the impact of foresight, cannibalization and market expansion on the optimal pricing strategy.

3.1. Demand model

Contemporary literature has considered changes in demand as being caused by "demand learning effects" such as consumption experience or positive word of mouth and "demand negative effects" such as market saturation. An interesting feature of the high-technology product market is that the demand structure apart from being influenced by these factors is also affected by changes caused due to the introduction of a more advanced version of the product. The second-generation product by virtue of its technological superiority widens the market base due to its added application potential. Furthermore, the introduction of the second-generation product provides an opportunity for potential buyers of the first-generation product to switch to the more advanced technology. This does not imply that the market for the older technology disappears overnight. There will be some consumers who still prefer the older technology. We incorporate all these aspects of the diffusion process in our demand model.

In keeping with tradition, we model the impact of demand effects on sales through cumulative sales (Robinson and Lakhani, 1975; Kalish, 1983, 1985, 1988). Since this experience can have a positive or negative effect on future sales, we do not make any assumptions regarding its direction. Instead, we study the effect of this on the optimal pricing policy.

Let \( x_i(t), i = 1, 2 \), denote the experience at time \( t \) (i.e., cumulative sales of product \( i \) up to time \( t \), \( i = 1, 2 \)). Then the derivative of experience with respect to time yields the instantaneous sales at time \( t \). We will denote this sales rate by \( \dot{x}_i(t) \) and this is a function of cumulative sales and price. Sales of the first-generation product for the interval before the introduction of the second-generation product is represented by the following demand equation

\[
\dot{x}_i(t) = f(x_i(t), p_i(t)).
\]

We do not ascribe any specific functional form for the demand equation in this section. This allows us to obtain the most general results. We will investigate specific cases in later sections to obtain stronger results. Let \( c_i = c_i(x_i(t)) \) denote production cost. We do not make any assumptions regarding the relationship between production costs of the two products. The assumption of experience curve effects requires that \( dc_i/dx_i < 0 \). We assume \( f_{p_1} = df(x_1, p_1)/dp_1 < 0 \). Subscripts will denote partial derivative with respect to that variable.

The introduction of the second-generation product at \( t = E \) widens the potential market due to its greater application potential. In addition, it also provides an opportunity for technological substitution by potential buyers of the earlier product. We propose to capture these effects by modelling the demand for the two products as follows. We use a parameter, \( \theta \), (where \( 0 \leq \theta < 1 \)) to capture the cannibalization effect. The sales equation for the first product for all time \( t > E \) is modified to

\[
\dot{x}_i(t) = (1 - \theta)f(x_i(t), p_i(t)).
\]
Market potential for the first-generation product:

\[ N \]

Sales of first-generation product:

\[ \dot{x}_1(t) = f(x_1, p_1) \]

Second-generation product introduced at \( t = E \)

Sales of first generation product:

\[ \dot{x}_1(t) = (1 - \theta)f(x_1, p_1) \]

Market potential for the second-generation product:

1. Switchers: \( \theta f(x_1, p_1) \)
2. Market for other uses: \( M \)

Sales of second generation product:

\[ \dot{x}_2(t) = g(x_1, p_1, x_2, p_2) \]

Fig. 2. The market description

The fraction of consumers who switch from the first-generation product are added to the potential market for the second-generation product. Note that while the fraction of consumers who switch is treated as a constant, the actual number of consumers who switch every period will be different. The sales of the second-generation product is represented as

\[ \dot{x}_2(t) = g(x_1(t), x_2(t), p_1(t), p_2(t)) \].

The maximum achievable market potential for the second product is the sum of the market potential for the first product and the new market potential from additional applications. To summarize, the price and cumulative sales of the first product determines the number of potential adopters of the first product. After entry of the second product a fraction \( \theta \) of the potential adopters of the first product switch to the second-generation product. The model assumes that the price of the second product does not affect the sales of the first product. The econometric study by Norton (1986) of the semi-conductor industry validates this assumption. The market description is as shown in Fig. 2.
3.2. Firm's problem

The firm seeks to maximize its total profits over the planning horizon \([0, T]\). The problem can be formalized as

\[
\max_{p_1, p_2} \int_0^T e^{-rt} \left[ (p_1(t) - c_1(t)) \dot{x}_1(t) + (p_2(t) - c_2(t)) \dot{x}_2(t) \right] \, dt,
\]

subject to

\[
\dot{x}_1(t) = (1 - \theta) f(x_1(t), p_1(t)), \quad \forall t,
\]

\[
\dot{x}_2(t) = g(x_1(t), x_2(t), p_1(t), p_2(t)), \quad \forall t \geq E,
\]

where

\[
\theta = \begin{cases} 
0 & \text{if } t < E, \\
\theta & \text{else},
\end{cases}
\]

\[
x_1(0) = x_{10}, \quad x_2(t) = 0 \quad \forall t \leq E.
\]

Henceforth, we eliminate the time argument for purpose of clarity. Note that the objective function is discontinuous at \(t = E\), the time of entry of the second product. We therefore solve the firm's optimization problem by breaking it up into two stages. The second stage focuses on the profit maximization problem subsequent to entry of the second product (i.e., \(t \geq E\)). The stage-two problem is

\[
\max_{p_1, p_2} \int_E^T e^{-rt} \left[ ((p_1 - c_1) \dot{x}_1 + (p_2 - c_2) \dot{x}_2) \right] \, dt,
\]

subject to

\[
\dot{x}_1 = (1 - \theta) f(x_1, p_1),
\]

\[
\dot{x}_2 = g(x_1, x_2, p_1, p_2),
\]

where

\[
x_1(E) = x_{1E}, \quad x_2(E) = 0.
\]

The solution yields the optimal prices \(p_1^*(t)\) and \(p_2^*(t)\) \(\forall t \in [E, T]\). The first stage then obtains the profit maximizing price for the first product for all time periods prior to entry of the second product assuming that optimal policies obtained in stage-two are adopted \(\forall t \in [E, T]\). The stage-one problem is

\[
\max_{p_1} \int_0^E e^{-rt} \left[ (p_1 - c_1) \dot{x}_1 \right] \, dt + J_2,
\]

subject to

\[
\dot{x}_1 = f(x_1, p_1),
\]

where

\[
x_1(0) = 0.
\]

Both stage-one and stage-two problems are optimal control problems and can be solved by invoking the Maximum Principle (see Kamien and Schwartz, 1981 for a review of optimal control theory). We assume that interior solutions to the problem exist. The analysis of the stage-two problem follows.
3.3. Stage two

The optimality conditions yield the optimal prices as

\[
p_1 = \frac{\eta_1}{\eta_1 - 1} \left[ c_1 - \lambda - \left( \frac{g_p}{f_p} \right) \left( \frac{p_2}{(1 - \theta)\eta_2} \right) \right],
\]

\[
p_2 = \frac{\eta_2}{\eta_2 - 1} [c_2 - \mu],
\]

where \( \lambda \) and \( \mu \) are the auxiliary variables associated with the two state variables \( \dot{x}_1(t) \) and \( \dot{x}_2(t) \), respectively, and \( \eta_i, i = 1, 2 \), are the elasticities of demand.

These results are generalizations of the myopically optimal pricing rule (marginal revenue = marginal cost) for a monopolist. The prices here incorporate not only the adjustment due to consideration of endogenous changes in the model but also an adjustment due to the effect of cannibalization. Comparison of prices of the two products are not possible without the use of very restrictive assumptions on costs and demand function specifications. We do not concern ourselves with this issue here.

The impact of cannibalization on the price of the first product depends on the substitutability between the two products (i.e., \( \theta \)) and on the ratio of sensitivity of sales of the two products to price of the first product. The impact of technological substitution alone on the price of the first product increases as the cannibalization increases and in the limit as \( \theta \) approaches one the optimal price gets driven down to marginal cost. Therefore, as substitution increases, the firm starts following a policy that closely resembles the myopic policy with respect to the first product.

If sales of the second-generation product are not affected by the price of the first product (i.e., \( g_{p_1} = 0 \)), then cannibalization has no effect at all on the optimal price of the first product. If sales of the second product increase in the price of the first product (i.e., \( g_{p_1} > 0 \)), then the effect of explicit consideration of cannibalization in the pricing problem is to cause the first-generation product to be priced higher. The firm in this situation prefers a strategy that limits sales of the first product with the use of a high price to a strategy that attempts to sell as much of the first product as possible after introduction of the more advanced version. On the other hand, if \( g_{p_1} < 0 \) then the effect of consideration of cannibalization in the pricing problem is to reduce the the price of the first-generation product. By doing so the firm moves as much of the first product as possible after introduction of the more advanced product. Cannibalization does not have a direct effect on the price of the second product but has an indirect effect through its impact on the auxiliary variable \( \mu \).

We now discuss the impact of endogenous changes and cannibalization on the auxiliary variables and its implications for the optimal pricing policy. The discussion on auxiliary variables hinges on how their behavior helps contrast the single-period myopic price with the dynamic price at each instant of time. The solution of the differential equations due to the multiplier conditions yields the following equation for the auxiliary variables

\[
\lambda(t) = -\int_t^T e^{-r(t-\tau)} \left[ c_1'(x_1)(1-\theta)f - (1-\theta) \frac{p_1 f_{x_1}}{\eta_1} + \frac{g}{g_{p_1}} \left( g_{x_1} - \frac{g_{p_1} f_{x_1}}{f_{p_1}} \right) \right] d\tau,
\]

\[
\mu(t) = -\int_t^T e^{-r(t-\tau)} \left[ c_2'(x_2) \dot{x}_2 - \frac{p_2 g_{x_2}}{\eta_2} \right] d\tau,
\]
where
\[ c'_i(x_i) = \frac{dc_i(x_i)}{dx_i}. \]

As can be seen from (7), the auxiliary variable is a combination of the cost decline as a result of an added unit of product sale, the revenue impact of that unit and the effect of cannibalization. Note that \( c'_i(x_i), i = 1, 2, \) is always negative (due to experience curve effects), implying that cost declines make the auxiliary variable positive and hence reduce price below cost or the myopic price. Demand learning effects like positive word-of-mouth or diffusion effects (i.e., \( f_x > 0, g_x > 0, g_x > 0 \)) create positive revenue effects and make it optimal to reduce price below cost while demand-negative effects like saturation or negative word of mouth create negative revenue effects which result in prices above cost.

The impact of cannibalization alone on the behavior of the auxiliary variable depends on the difference in the demand effect (of an added unit of sale of the first product) on the second product versus the first after controlling for sensitivity of sales of the two products to the price for the first product. To understand this better, consider the following extreme situations. Consider a case where there is no experience curve effect. Now if an added unit sale of the first product has a greater effect on decreasing the total market size of the second product than the first, then \( \lambda(t) < 0, \) implying that it is optimal for the firm to price the first product above cost. This makes intuitive sense. If every added unit sale of the first product not only reduces the market size of the first product due to saturation but also that of the second, it makes sense for the firm to charge as high a price for the first product as possible. If now we add experience curve effects, the firm faces a trade-off between wanting to price above cost for reasons cited above and to price below cost to sell more and come down the experience curve faster. The actual sign of the auxiliary variable will now depend critically on the relative magnitudes of these effects. In contrast, situations where an added unit of sale of the first product has a greater effect on increasing the total market size for the second product than the first (after controlling for price sensitivity) render the auxiliary variable \( \lambda(t) > 0, \) implying that it is optimal for the firm to price below cost. The firm, in effect, subsidizes the early buyers since that will create a greater demand for both products in the future due to demand learning effects. Since \( \mu(t) \) is also positive in the presence of demand learning effects the price of the second product will also be below its cost. We summarize the results of the cross-price effects and the cross-demand effects of cannibalization on the price of the first product in Table 1. We now discuss the impact of the degree of substitution on the magnitude of this effect.

**Proposition 1:** As the technological substitution increases, \( \lambda(t) \) tends to zero and the optimal price of the first product approaches the myopic price (proof in appendix).

<table>
<thead>
<tr>
<th>Cross-price effect</th>
<th>Cross-demand effect</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g_{p_1} &gt; 0 )</td>
<td>( g_{x_1} &lt; 0 )</td>
<td>( p_1 &gt; c_1 )</td>
</tr>
<tr>
<td>( g_{p_1} &lt; 0 )</td>
<td>( g_{x_1} &gt; 0 )</td>
<td>( p_1 &lt; c_1 )</td>
</tr>
</tbody>
</table>

Here \( g_{p_1} = \frac{dg}{dp_1}, g_{x_1} = \frac{dg}{dx_1}. \)
In other words, as cannibalization increases, the marginal value of an added unit of sale of the first product on total profit goes to zero. Recall that if the diffusion effect is uniformly positive, then \( \lambda(t) > 0 \) (i.e., the sale of an additional unit of the first product has a positive impact on total profits). The proposition implies that this value of an added unit of sale of the first product on total profit decreases with increase in cannibalization and in the limit (i.e., \( \theta = 1 \)) is equal to zero. If \( f_{x_1} < 0 \), then \( \lambda(t) < 0 \) and the proposition implies that, as cannibalization increases, the negative effect of an added unit of sale of the first product actually decreases until in the limit it stops having any effect at all on total profit. In both cases, the impact of cannibalization is to drive prices towards the myopic price.

The optimal policy for the second product requires that with demand learning effects the deviation from the myopic price increases as cannibalization increases. Overall, as substitution increases, the price of the first product gets closer to its myopic price, while the price for the second product gets farther away from its myopic price. The other question of interest in situations like these is the behavior of price over time. To answer that question, it is necessary to obtain the time derivative of the optimal price given by (5) and (6). The second-order condition for maximization of profits leads to the following condition for price path behavior of the second product:

\[
\dot{p}_2 = \text{Sign} \left[ -ru + \frac{g_{x_2} p_2}{\eta_2} + F \dot{x}_2 \right],
\]

(8)

where

\[
F = \left[ \frac{gg_{p_2 x_2} - g_{p_2} g_{x_2}}{g_{p_2}^2} \right].
\]

As can be seen from (8), the behavior of price over time is affected by factors such as the interest rate, demand effect and other second order effects. The basic insights are identical to those of Kalish (1983). Consequently, we state the following results without proof. Proofs follow the same approach as in Kalish (1983).

**Proposition 2:** With experience curve effects alone, the price of the second-generation product is monotonically decreasing over time.

**Proposition 3:** In the absence of discounting, price of second-generation product is monotonically increasing when \( g_{x_2} > 0 \) and \( g_{p_2 x_2} > 0 \).

**Proposition 4:** In the absence of discounting, the price of the second-generation product is monotonically decreasing when \( g_{x_2} < 0 \) and \( g_{p_2 x_2} < 0 \).

**Proposition 5:** For the case of a separable demand function with experience curve and positive discount rate, if \( g''(x_2) > 0 \), then the following characterize the optimal price policy for the second-generation product.

i. If \( g'(x_2(T)) < 0 \), then \( \dot{p}_2 < 0 \ \forall t \in [E, T] \).

ii. If \( g'(x_2(E)) < 0 \) and \( g'(x_2(T)) > 0 \), then the price is decreasing and then increasing.

iii. If \( g'(x_2(E)) \) then price is either monotonically increasing or decreasing and then increasing.

\[1\] The proofs are available from the first author upon request.
It is difficult to obtain identical propositions for the first-generation product due to the complicated structure of (4). To summarize, the results from analysis of the stage-two problem indicate that while earlier finding of Kalish (1983) and others regarding price path behavior are preserved in our model, the actual prices changed at each instant of time for the two products are significantly different with the consideration of product line issues in the profit maximization problem.

3.4. Stage one

We have obtained from our analysis of the stage-two problem the optimal prices \( p_1(t) \) and \( p_2(t) \) \( \forall t \in [E, T) \). The second-stage problem is now an optimal control problem with salvage value and can be solved using standard techniques. The optimality condition yields

\[
p_1 = \frac{\eta_1}{\eta_1 - 1} (c_1 - \beta),
\]

where \( \beta \) is the auxiliary variable and can be obtained by solving the multiplier equation

\[
\frac{d\beta}{dt} = r\beta - \frac{dH_2}{dx_1},
\]

s.t. \( \beta(E) = \frac{dJ_2}{dx_1} \).

The solution to the differential equation yields

\[
\beta(t) = e^{-r(E-t)}\beta(E) - \int_t^E e^{-r(\tau-t)} \left[ \frac{f_f}{f_p} - c_1 \right] d\tau.
\]

Note that apart from usual considerations regarding the impact of cumulative sales and cost on the behavior of the auxiliary variable (and consequently the price of the first product), the change in the salvage value with respect to an added unit of sale of the first product also affects the behavior of the auxiliary variable. The impact of foresight is to increase the deviation from the myopic price obtained through considerations of endogenous demand effects by an amount equal to \( \exp(-r(E-t))\beta(E) \). As expected, increase in interest rate decreases the impact of foresight on the behavior of the auxiliary variable.

It is easy to show that with demand learning effects the sale of an added unit of the first product in pre-entry phase (stage one) increases the total profit post-entry (i.e., \( \beta(E) > 0 \)). The intuition here is that an added unit of sale of the first product increases total market size for the first product and also increases the potential market for the second product from cannibalized sales. Consequently, the impact of foresight is to drive down the price of the first product below marginal cost in the pre-entry phase. The integral term in (11), which refers to the impact of endogenous demand changes, is negative and serves to accentuate the drop in price caused by consideration of foresight alone. As a result, the price of the first product will be always below cost. Note that the auxiliary variable is positive \( \forall t \in [0, E) \) and for the case of zero discount rate is monotonically decreasing to \( \beta(E) \).

In contrast, with market saturation effects the sale of an added unit of sale of the first product in the pre-entry phase has an adverse effect on post-entry profits (i.e., \( \beta(E) < 0 \)). The
intuition here being that every extra unit of product sold reduces the size of the available market at the time of entry of the more advanced version. The consideration of foresight therefore leads to the firm adopting a strategy of pricing above marginal cost. Note that this also serves the benefit of limiting market penetration of the first product till the time of entry of the second product. Again, the consideration of endogenous demand effects is to accentuate this effect so that the price of the first product will be further driven up above cost and hence the observed phenomenon of higher margins in early periods with the knowledge of impending entry of a more advanced version of the product at time \( t = E \). Note that the sign of \( \beta(E) \) depends only on the nature of the demand effect. The above results are summarized in Table 2.

The presence of the salvage value term in the stage-one problem has no impact on the price path behavior of the first product. In other words, while the actual level of price at every instant of time is dependent on it, the price path behavior is not. Consequently, the standard results for price path behavior emerge and we do not discuss them here.

4. Integrated monopolist vs independent producers

We have characterized the optimal price policy for a firm that produces both the first- and second-generation product. We will refer to this firm as the integrated monopolist. It is quite conceivable that there are situations where the two products are introduced by two different firms. The copier industry where Xerox followed 3M by developing a better copying process, the desk-top publishing industry where Apple followed Xerox, etc. are some examples of this kind of situation. As Kotler (1991) points out, “Technological leapfrogging is a bypass strategy often used in high-tech industries”.

Let these producers be referred to as independent producers. We now seek to contrast the optimal pricing strategy of the integrated producer to the pricing strategy of the independent producers. In other words, all other things being constant, how does the fact whether the same firm produces both products or not affect the pricing strategy for the two products. To focus on this aspect, we assume that the entry time for the second-generation product is the same (i.e., at \( t = E \)) and there is no latitude with respect to this variable for both the integrated and independent producer. This avoids possible distortions in policy caused due to gaming by producers on the timing of product entry. Let the superscript \( i \) denote the independent producers.

The profit maximization problem for the independent monopolist producing the first-generation product is

\[
\max_{p_1^i} R = \int_0^T e^{-rt} [p_1^i - c_1^i] \dot{x}_1^i(t) \, dt, \\
\text{s.t. } \dot{x}_1^i = (1 - \theta) f(x_1^i, p_1^i),
\]

\[(12)\]
where
\[
\theta = \begin{cases} 
0 & \text{if } t < E, \\
\theta & \text{else,}
\end{cases} \quad x_i^1(0) = 0.
\]

The discontinuity at \( t = E \) caused by introduction of the second-generation product is handled as before by breaking the problem into two stages.

4.1. Stage two

The stage-two profit maximization problem for the independent producer of the first product is
\[
\max_{p_1^i} R_1 = \int_E^T e^{-rt} \left[ p_i^1 - c_i^1 \right] x_1 \, dt, \tag{13}
\]
\[
s.t. \quad \dot{x}_1^i = (1 - \theta) f(x_1^i, p_1^i),
\]
\[
\text{where} \quad x_1^i(E) = x_{1E}^i.
\]

The manufacturer of the second-generation product solves his profit maximization problem using the price of the first product as given. In other words, the producer of the second-generation product is a Stackelberg follower. His problem can be formalized as
\[
\max_{p_2^i} R_2 = \int_E^T e^{-rt} \left[ p_2^i - c_2^i \right] x_2 \, dt, \tag{14}
\]
\[
s.t. \quad \dot{x}_2^i = g(x_1^i, x_2^i, p_1^i, p_2^i),
\]
\[
\text{where} \quad x_2^i(E) = 0.
\]

For the sake of analytical tractability, we will use a specific functional form. The specification used here is similar to the model used by Rao and Bass (1985). Price is modeled as having a multiplicative effect on adoption and we allow for market saturation effects. The model is represented as
\[
x(t) = f(x, p) = (N - x) e^{-kp},
\]
where \( 0 < k < 1 \).

We will also assume that cost of production is constant and identical for the integrated and independent producer for each of the two products. We do so because we wish to focus solely on the differential impact of foresight and the cannibalization and market expansion effects on the optimal pricing policies of the independent and integrated producer. Let \( N \) denote the market potential for the first-generation product and \( M \) denote the market potential of new applications of the second-generation product. The demand model for the two products is then characterized by the following set of equations
\[
\dot{x}_1^i(t) = (1 - \theta)(N - x_1^i) e^{-k_1 p_1^i},
\]
\[
\dot{x}_2^i(t) = [\theta(N - x_1^i) e^{-k_1 p_1^i} + M - x_2^i] e^{k_2 p_2^i}. \tag{15}
\]
It is easy to show that the qualitative nature of results do not change if we assume that the price sensitivity of the two products is identical (i.e., \( k_1 = k_2 = k \)). We will use this simplification in subsequent analysis. Solution of the optimization problem using standard techniques leads to the following theorem that formalizes the relationship between prices of the two producers for the first product for \( t \in [E, T] \).

**Theorem 1:** The integrated producer has a lower price for the first-generation product than the independent producer for \( t \in [E, T] \) (proof in appendix).

The intuition is simple. The integrated producer is more concerned with overall profits. For the integrated producer a low price on the first product increases the potential sales and that leads to an increase in the sales of the second product due to cannibalization. He can therefore make up any lost sales on the first by the cannibalized sales of the second product. The independent producer on the other hand does not have that luxury. It is therefore optimal for him to limit the sales of the first product, which limits the cannibalization, and he achieves this by following a strategy that consistently charges a higher price for the first-generation product after entry of the newer technology.

It is easy to show that the marginal value of an added unit of sale of the first product is lower for the integrated producer than the independent producer. The result is driven by the fact that an added unit of sale of the first product impacts adversely on future potential sales of both products for the integrated producer whereas for the independent producer the impact is only with respect to the first product. This implies that consideration of dynamic endogenous changes alone would have led to a reversal of Theorem 1. It is the consideration of the substitution phenomenon (i.e., cannibalization effect) in the overall profit maximization problem that causes the observed differences in pricing strategy between the two producers.

**Theorem 2:** Both producers charge the same price for the second-generation product (proof in appendix).

The result is driven by the fact that while the price of the second generation impacts the price path behavior of the first, the reverse does not hold true. Consequently, even though both producers face different strategies for the first generation, that does not impact on the pricing strategy for the second product. The profit maximization problem for both producers yield identical differential equations that have similar boundary conditions and hence it is not surprising that the pricing behavior is similar. Interestingly, this does not imply that they sell identical amounts of the second-generation product. The difference in sales is created by the differences in pricing strategies of the producers of the first product. That completes the comparison of stage-two policies and we now proceed to characterize the optimal strategies in the second stage.

### 4.2. Stage one

Recall that \( \beta(E) < 0 \) since we have market saturation effects in the demand model. We begin by formalizing the relationship between the price of the first product immediately prior to entry of the second-generation product denoted by \( p_1(E^-) \) to its price after entry denoted by \( p_1(E^+) \).
Proposition 6: \( p_t(E^-) > p_t(E^+) \) (proof in appendix).

Since prices are monotonically declining over time, this implies that the price for the first-generation product for all periods prior to entry is higher than its price after entry of the second product. It is easy to show that a similar relationship between the prices is also obtained for the independent monopolist.

Proposition 7: \( p_1(E^-) > p_1(E^+) \) (proof in appendix).

We have already shown that \( p_1 < p_1^* \) for all time periods after the entry of the second product. The solution of the optimization problem for the two producers leads to the following theorem that contrasts their policies with regard to the first product in the pre-entry stage. The price path behavior over time of the first product for the two producers is shown in Fig. 3.

Theorem 3: \( p_1(t) > p_1^*(t) \ \forall t \in [0, E) \) (proof in appendix).

In other words, the integrated producer charges a higher price for the first product in all periods prior to entry of the second product. As the proof indicates, this relationship between prices is essentially driven by the differences in post-entry profits (caused by an additional unit of sale of the first product in the pre-entry stage) of the two producers. The intuition here is that in the presence of market saturation effects, an extra unit of first product sale in the pre-entry stage implies two different things for the two producers. For the independent producer it implies that this unit will no longer by cannibalized by the second product. For the integrated producer it translates to reducing the cannibalizable potential market for the second...
product. The mathematical translation of this insight is reflected by the observation that an added unit of sale of the first product in pre-entry periods has a greater negative impact on total post-entry profits for the integrated producer than the independent producer. Consequently, the integrated producer finds it optimal to price the first product higher and make as great a margin as possible on the units sold whereas the independent producer finds it optimal to price low and sell as much as possible before \( E \). After the entry of the second-generation product, the relationship between the prices of the first product of the two producers flips over. Now an integrated producer gains more from a price decrease (due to increased market potential for the second caused by cannibalization) and therefore finds it optimal to charge a price lower than the independent producer. The intuition regarding the market penetration achieved by the independent producer of the first product relative to the integrated producer before entry of the new product is formalized in the following corollary.

**Corollary 1:** \( x_i(E) < x_j(E) \) (proof in appendix).

The independent producer by pricing lower in the pre-entry stage manages to “move” as much of the first-generation product as possible prior to the entry of the second-generation product. This strategy ensures that the competitive effect following introduction of the second product is minimized as much as possible. Following introduction of the second product, the independent producer finds it optimal to follow a strategy that ensures the greatest margins possible on units sold before the end of the planning horizon. That completes the comparison of the pricing strategies adopted by the two producers for the first product and the consequences (see Table 3 for a summary of the results). It should be pointed out once again that even though the producers follow identical pricing policies for the second product, the profits obtained from the second product sales are not identical since sales volumes are different.

A valid concern here is the robustness of the derived results regarding price comparisons to various functional specifications of the demand model. Comparison of the pricing policies of the two producers with demand models allowing for market growth effects provides an interesting contrast and some important insights. We used a demand model represented as \( f(x, p) = (a + bx) \exp(-kp) \). We can show that, in this situation, the independent producer will always charge a higher price for the first product compared to the integrated producer. The earlier policy of greater penetration of the market by the independent producer is sub-optimal in the presence of market growth effects because it leads to greater cannibalization in the post-entry phase. It is therefore optimal for the independent producer to follow a policy that maximizes on margins at every point in time. The independent producer will always charge a

<table>
<thead>
<tr>
<th>Table 3</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Comparison of price policies of integrated and independent producers</strong></td>
</tr>
<tr>
<td><strong>Stage 1.</strong></td>
</tr>
<tr>
<td>( p_i &gt; p_i^1 )</td>
</tr>
<tr>
<td>( x_i(E) &lt; x_i^1(E) )</td>
</tr>
</tbody>
</table>

Here \( p_i \) \( (j = 1, 2) \) = price charged by integrated producers,

\( p_i^1 \) \( (j = 1, 2) \) = price charged by independent producers,

\( x_i \) \( (j = 1, 2) \) = cumulative sales of integrated product,

\( x_i^1 \) \( (j = 1, 2) \) = cumulative sales of independent producer.
higher price and sell less of the first product than the integrated producer. Viewed from the perspective of the integrated producer, this policy in the pre-entry stage makes sense because the presence of market growth effects and a finite time horizon dictate that he price the product so as to achieve the greatest penetration. That will translate to an increased market potential for the second product from product substitution.

Our intuition based on these results suggests that reversals of the relationships between prices charged by the two producers in the pre-entry and post-entry phase will occur in all models that consider market saturation. The reason for this contention is based on the fact that saturation causes the independent producer to want to move as much of the first product as possible in the pre-entry stage. The integrated producer prefers to limit this penetration since it decreases the potential gains for the second product. Given this dichotomy in objectives, it makes sense that the independent producer prefers to price lower than the integrated producer in the pre-entry periods. Since it is reasonable to assume that market saturation is a more realistic scenario than unlimited market growth, we believe that price reversal effects obtained earlier are more relevant. It does seem that the robustness of the results have more to do with the specific nature of the demand effect rather than the exact specification of it.

5. Conclusion

We analyze the impact of endogenous changes, technological substitution and foresight on the optimal dynamic pricing policy of a firm marketing successive generations of product advances. We show that consideration of cannibalization, market expansion and foresight leads to significantly different prices for the product innovations. We discuss the policy implications of these considerations with regard to pricing of successive product advances in the following paragraphs.

Recall that the price that maximizes immediate profits for a monopolist is the price that equates marginal revenue with marginal cost. The consideration of factors such as discounting and endogenous changes in demand alter the nature of that price. The optimal price now incorporates an adjustment that captures the future marginal benefit of an added unit of sale. In our model, the price of the product apart from incorporating this also includes an adjustment for the cannibalization caused by introduction of the second generation product. The direction of this effect (i.e., increase or decrease in price) depends on the effect of the first product's price on sales of the improved product and on the technological substitution between the two products. If sales of the more advanced product increase with an increase in price of the first product, then it is optimal for the firm to charge a higher price for the first product even though that strategy limits the sales of the product. If sales of the second generation product increase with a decrease in price of the first product, then it is optimal for the firm to drop its price for obvious reasons. In the absence of any sensitivity of second product's sales to the price of the first product, the optimal prices are same as the standard dynamic optimal price. The overall effect of technological substitution is to drive the price of the first product towards the myopic policy as substitution increases.

The consideration of foresight also alters the price from that obtained with consideration of endogenous demand effects done. The change in price is by an amount equal to the change in total profit after introduction of the second product caused by an added unit of sale of the first
product in the pre-entry stage. With demand learning effects (like positive word-of-mouth etc.), this marginal benefit is positive and the price for the first product in pre-entry periods is lower with the consideration of foresight. With demand negative effects (e.g., saturation effects), the reverse holds and the price of the first product is higher in pre-entry periods with the consideration of foresight. The results also indicate that the intuition of Kalish (1983) with respect to price path behavior over time are preserved in our model.

We make the assumption in our model that there are no other competitors in the market. The demand for the product depends on the monopolist’s price alone. Our model would serve as a reasonable approximation of the market in situations where there exist barriers to entry due to factors such as patent protection or proprietary nature of technology. The successful lawsuits settlements in favour of Texas Instruments, Intel, etc., seem to indicate that there does exist some level of entry barrier in the technology market. To the extent that they do not exist, the intuition of our model would be moderated by competitive pressures. We also make the assumption that $\theta$, which represents the degree of technological substitution, is constant over time. It is plausible that $\theta$ may vary in some systematic manner over time. In fact, an argument could be made for $\theta$ increasing over time based on increasing sophistication of the market and its capacity to absorb the newer technology. It is possible to show that this effect alone creates a decreasing trend in the price of the first product over time and an increasing trend in the price of the second product over time. In other words, increasing cannibalization over time leads the firm to adopt a skimming policy for the older technology and a penetration policy for the newer technology. Obviously, the actual price path behavior would depend on the relative magnitude of this effect with the effects of discount rate, endogenous demand effects and other second order effects.

We also analyze the pricing problem for the case where the successive product innovations are introduced by different firms or what we refer to as independent producers. The key result here is that the price charged for the first product over the planning horizon $[0, T]$ are significantly different depending on whether it is produced by an integrated monopolist or an independent producer. Before the entry of the product innovation, the integrated monopolist charges a higher price for the first product than the independent producer. As a consequence, the market penetration achieved by the two firms for the first product by the time of entry is different. The independent producer achieves a greater penetration of the market before the entry of the second product. After entry of the second product, the relationship between the prices charged by two producers changes. Technological substitution in this stage causes the integrated monopolist to charge a lower price for the first product compared to the independent producer.

The demand model used in the analysis assumes that the time of entry of the second product is determined exogenously. The results from a numerical simulation of the basic pricing model (details available upon request) with varying entry times for the second generation product indicate that it is not optimal for the integrated firm to delay entry of the more advanced version of the product. These results hold for a considerable range of parameter values and are also consistent with the findings of Wilson and Norton (1989). Our conjecture is that endogenous consideration of entry timing will extenuate the tendency of the independent producer of the earlier technology to increase market penetration of the product. The independent producer of the newer technology will prefer earlier entry. As the model in its present version does not include factors such as higher R&D costs for faster introduction, such a result would not be surprising. Future research attempting analytical consideration of this problem should
incorporate such issues in the formulation. Endogenous consideration of this issue would be a very worthwhile contribution to the literature.

**Appendix**

The demand model is

\[
\begin{align*}
\dot{x}_1(t) &= (1 - \theta) f(x_1, p_1), \\
\dot{x}_2(t) &= g(x_1, p_1, x_2, p_2).
\end{align*}
\]

The current value Hamiltonian to this problem is

\[
H = (p_1 - c_1 + \lambda)(1 - \theta) f(x_1, p_1) + (p_2 - c_2 + \mu) g(x_1, x_2, p_1, p_2).
\]

The requirement of optimality of prices for the two products at every instant of time (i.e., \(dH/dp_1 = 0, dH/dp_2 = 0\)) implies that

\[
\begin{align*}
(p_2 - c_2 + \mu) g_{p_2} + g &= 0, \\
(p_1 - c_1 + \lambda)(1 - \theta) f_{p_1} + (1 - \theta) f + (p_2 - c_2 + \mu) g_{p_1} &= 0.
\end{align*}
\]

After some algebra, these equations yield the optimal prices as reported in equations (4) and (5) in the text. The multiplier equation obtains the following differential equation

\[
\frac{d\lambda}{dt} = \dot{\lambda} = r\lambda - \frac{dH}{dx_1} \\ = r\lambda - \left[(p_1 - c_1 + \lambda)(1 - \theta) f_{x_1} - c_1'(x_1)(1 - \theta) f + (p_2 - c_2 + \mu) g_{x_1}\right],
\]

subject to the boundary condition \(\lambda(T) = 0\). The solution to this yields

\[
\lambda(t) = \int_t^T e^{-r(t-\tau)} \left[\frac{(1 - \theta) f_{x_1}}{f_{p_1}} + c_1'(x_1)(1 - \theta) f + \frac{g}{g_{p_2}} \left[g_{x_1} - \frac{g_{p_1} f_{x_1}}{f_{p_1}}\right]\right] d\tau.
\]

A similar procedure obtains the solution to the other auxiliary variable as

\[
\mu(t) = -\int_t^T e^{-r(t-\tau)} \left[g_{x_2} + c_2'(x_2) g\right] d\tau.
\]

**Proof of Proposition 1**

Recall that \(\theta\) is the constant that captures the degree of cannibalization. Differentiation (A-3) with respect to \(\theta\) obtains

\[
\frac{d\lambda}{d\theta} = \int_t^T e^{-r(t-\tau)} \left[ff_{x_1} + c_1'(x_1)f\right] d\tau.
\]

Note that \(\lambda(t) > 0\) requires \(f_{x_1} > 0\). Since \(c_1'(x_1) < 0\) and \(f_{p_1} < 0\), that implies \(d\lambda/d\theta < 0\). Similarly, \(\lambda(t) < 0\) requires \(f_{x_1} > 0\) and \(c_1'(x_1) = 0\). In that situation we have \(d\lambda/d\theta < 0\). Also, \(\lambda(t) = 0\) when \(\theta = 1\). Hence the proposition.
Proof of Theorem 1

For the sake of easier exposition, we initially prove the theorem for the case where \( r = 0 \). The multiplier equation for integrated monopolist's optimization problem yields

\[
\frac{d\lambda}{dt} = (1 - \theta) \frac{e^{-kP_1}}{k} > 0, \tag{A-6}
\]

s.t. \( \lambda(T) = 0 \).

The optimal prices for the integrated monopolist obtained from the solution of the optimization problem are

\[
p_1(t) = c_1 - \lambda(t) + \frac{1}{k} - \frac{\theta e^{-kP_2(t)}}{(1 - \theta)}, \tag{A-7}
\]

\[
p_2(t) = c_2 - \mu(t) + \frac{1}{k}.
\]

Taking derivatives with respect to time obtains the equation for price path behavior as

\[
\frac{dp_1(t)}{dt} = \dot{p}_1(t) = -\dot{\lambda}(t) + \frac{\theta e^{-kP_2}}{(1 - \theta)}, \tag{A-8}
\]

\[
\frac{dp_2(t)}{dt} = \dot{p}_2(t) = -\dot{\mu}(t) < 0. \tag{A-9}
\]

Therefore, prices of both products are monotonically decreasing with time. A similar procedure with respect to the independent producer yields

\[
\frac{dp_1^i(t)}{dt} = \dot{p}_1^i(t) = -\dot{\lambda}_i(t) = -(1 - \theta) \frac{e^{-kP_1^i(t)}}{k} < 0 \tag{A-10}
\]

Suppose there exists a time \( t^* \) such that \( p_1(t^*) > p_1^i(t^*) \). We show that this implies \( \dot{p}_1(t^*) > \dot{p}_1^i(t^*) \) and furthermore this holds for all \( t > t^* \). That implies \( p_1(T) > p_1^i(T) \), which is a contradiction, and hence the theorem. We have from (A-8) and (A-10),

\[
\dot{p}_1 - \dot{p}_1^i = \frac{(1 - \theta)}{k} \left[ e^{-kP_1} - e^{-kP_1^i} \right] + k P_2 \frac{\theta e^{-kP_2}}{1 - \theta}, \tag{A-11}
\]

since \( p_1 > p_1^i \), we have \( \exp(-kP_1) < \exp(-kP_1^i) \). Furthermore, \( \dot{p}_2 < 0 \), which implies that \( \dot{p}_1 < \dot{p}_1^i \). Therefore the price paths of the two products will never intersect and hence the theorem.

For the more general case of non-zero interest rate, the equations governing the price path behavior of the first product for the integrated and independent producer are

\[
\dot{p}_1 = r \left[ p_1 + \frac{\theta e^{-kP_2}}{1 - \theta} - c_1 + \frac{1}{k} \right] + \frac{k \theta \dot{p}_2 e^{-kP_2}}{1 - \theta} - \frac{(1 - \theta) e^{-kP_1}}{k}, \tag{A-12}
\]

\[
\dot{p}_1^i = r \left[ p_1^i - c_1 - \frac{1}{k} \right] - \frac{(1 - \theta) e^{-kP_1^i}}{k}. \tag{A-13}
\]
Therefore,

\[ \dot{p}_1 - \dot{p}_1^i = r[p_1 - p_1^i](1 - \theta)[e^{-kp_1^i} e^{-kP_1}] + \frac{\theta e^{-kp_2}}{1 - \theta}[1 + k\dot{p}_2]. \]  

(A-14)

Consequently, \( p_1(t^*) > p_1^i(t^*) \Rightarrow \dot{p}_1 > \dot{p}_1^i \) and that leads to a contradiction since it implies \( p_1(T) > p_1^i(T) \). Hence the theorem.

\textit{Proof of Theorem 2}

The solution to the optimization problem of the integrated producer obtains the following equation for the price of the second product:

\[ p_2 = c_2 - \mu + \frac{1}{k}. \]  

(A-15)

Taking derivatives with respect to time, we obtain

\[ \dot{p}_2 = \frac{dp_2}{dt} = -\dot{\mu} \]

\[ = -r\mu - \frac{e^{-kp_2}}{k} \]

\[ = r \left( p_2 - c_2 - \frac{1}{k} \right) - \frac{e^{-kp_2}}{k}. \]  

(A-16)

Similarly, for the case of the independent producer of the second generation product, we obtain

\[ p_2^i = c_2 - \mu^i + \frac{1}{k}, \]

\[ p_2^i = r \left( p_2^i - c_2 - \frac{1}{k} \right) - \frac{e^{-kp_2}}{k}. \]

The differential equations for both producers and the boundary conditions are identical and so we obtain \( p_2^i(t) = p_2(t) \ \forall t \in [E, T] \); hence the theorem.

\textit{Proof of Proposition 6}

For all \( t < E \), we have

\[ p_1(t) = c + \frac{1}{k} + \beta(t) - e^{-r(E-t)}\beta(E). \]  

(A-17)

Therefore taking limits as \( t \to E^- \), we obtain

\[ p_1(E^-) = c + \frac{1}{k} - \beta(E). \]  

(A-18)

For all \( t > E \), we have

\[ p_1(t) = c + \frac{1}{k} + \lambda(t) - \frac{\theta e^{-kp_2}}{(1 - \theta)}. \]  

(A-19)
Therefore taking limits as $t \to E^+$, we obtain
\[ p_i(E^+) = c + \frac{1}{k} + \lambda(E) - \frac{\theta e^{-kp_2}}{(1 - \theta)}. \] (A-20)

On comparing (A-24) and (A-26), we obtain $p_i(E^-) > p_i(E^+)$ since $\theta e^{-kp_2}/(1 - \theta) > 0$, and as we show below $\beta(E) < \lambda(E)$.

Recall that
\[
\beta(E) = \frac{dJ_1}{dx_1} = -\int_{E}^{T} e^{-rt} \left[ \left((p_1 - c_1)(1 - \theta) + \theta(p_2 - c_2) e^{-kp_2}\right) e^{-kp_1} \right] d\tau
\]
\[
= -\int_{E}^{T} e^{-rt} \left[ \frac{(1 - \theta)(1 - \lambda - \mu)}{k} e^{-kp_1} \right] d\tau,
\] (A-21)

and
\[
\lambda(t) = -\int_{E}^{T} e^{-r(\tau-t)} \left[ \frac{(1 - \theta) e^{-kp_1}}{k} \right] d\tau.
\]

Since $\exp(-rt) > \exp[-r(\tau - t)]$ and $(1 - \lambda - \mu) > 0$, we have $\beta(E) < \lambda(E)$, hence the proposition.

**Proof of Proposition 7**

For all $t > E$, we have
\[ p_i(t) = c + \frac{1}{k} - \lambda(t). \] (A-22)

Therefore, taking limits as $t \to E^+$, we obtain
\[ p_i(E^+) = c + \frac{1}{k} - \lambda(E) \] (A-23)

Similarly for all $t < E$, we have
\[ p_i(t) = c + \frac{1}{k} - \beta(t) - e^{-r(e-t)}\beta(E). \] (A-24)

Therefore, taking limits as $t \to E^-$, we obtain
\[ p_i(E^-) = c + \frac{1}{k} - \beta(E). \] (A-25)

Comparing the two we obtain
\[ p_i(E^+) = p_i(E^-), \]
since we can show as in the previous proposition that $\beta(E) \leq \lambda(E)$, hence the proposition.
Proof of Theorem 3

We have

\[
p(t) = c - \beta(E) + \frac{1}{k} + \int_t^E e^{-k p(\tau)} \, d\tau,
\]

\[
p'(t) = c - \beta'(E) + \frac{1}{k} + \int_t^E e^{-k p'(\tau)} \, d\tau,
\]

\[
[p'(t) - p(t)] - \int_t^E [e^{-k p'(\tau)} - e^{-k p(\tau)}] \, d\tau = [\beta(E) - \beta'(E)]
\]

\[
\Rightarrow \text{Sign}[p'(t) - p(t)] = \text{Sign}[\beta(E) - \beta'(E)]
\]

\[
= \text{Sign} \left[ \frac{dJ_1}{dx_1} - \frac{dR_1}{dx_1} \right]
\]

\[
= \text{Sign} \left\{ e^{-rt}(1 - \theta) \left[ (p_1 c_1) e^{-k p_1} - (p_1 - c_1) e^{-k p_1} \right] \right. \\
\left. - \theta e^{-rt}(p_2 - c_2) e^{-k p_2} e^{-k p_1} \right\}
\]

\[
= \text{Sign} \left[ (p_1 c_1) e^{-k p_1} - (p_1 - c_1) e^{-k p_1} \right] - \frac{\theta}{1 - \theta} (p_2 - c_2) e^{-k p_2} e^{-k p_1}
\]

\[
= \text{Sign} \left[ (p_1 - c_1) e^{-k p_1} - (p_1 - c_1) \right] - \frac{\theta}{1 - \theta} (p_2 - c_2) e^{-k p_2}
\]

\[
= \text{Sign} \left[ p_1 - p_1 - \frac{\theta}{1 - \theta} (p_2 - c_2) e^{-k p_2} \right]
\]

\[
= \text{Sign} \left[ \lambda - \lambda' + \frac{\theta}{1 - \theta} e^{-k p_2} - \frac{\theta}{1 - \theta} (p_2 - c_2) e^{-k p_2} \right] < 0
\]

Since \( p < p^i \) for all \( t \in [E, T] \), we have \( \lambda < \lambda^i \). Also \( p_2 - c_2 = 1/k - \beta > 1 \) (since \( \beta < 0 \) and \( 0 < k < 1 \)). Hence \( p < p^i \) \( \forall \) \( [0, E] \).

Proof of Corollary 1

\[
\frac{dx}{dt} = (N - x) e^{-k p}
\]

\[
\Rightarrow \frac{dx}{N - x} = e^{-k p} \, dt
\]

\[
\Rightarrow x(t) = N \left( 1 - \exp \left[ \int_0^t e^{-k p(\tau)} \, d\tau \right] \right).
\]

Since \( p_1 > p^i \) \( \forall t \in [0, E] \), we have

\[
\int_0^t e^{-k p(\tau)} \, d\tau > \int_0^t e^{-k p(\tau)} \, d\tau,
\]

which implies \( x^i(E) > x^i(E) \).
References


